1. A group of n 2 people decide to play an exciting game of Rock-Paper Scissors. As you may recall, Rock smashes Scissors, Scissors cuts Paper, and Paper covers Rock (despite Bart Simpson saying “Good old rock, nothing beats that!”). Usually, this game is played with 2 players, but it can be extended to more players as follows. If exactly 2 of the 3 choices appear when everyone reveals their choice, say a, b 2 {Rock, P aper, Scissors} where a beats b, the game is decisive: the players who chose a win, and the players who chose b lose. Otherwise, the game is indecisive and the players play again. For example, with 5 players, if one player picks Rock, two pick Scissors, and two pick Paper, the round is indecisive and they play again. But if 3 pick Rock and 2 pick Scissors, then the Rock players win and the Scissors players lose the game. 1 Assume that the n players independently and randomly choose between Rock, Scissors, and Paper, with equal probabilities. Let X, Y, Z be the number of players who pick Rock, Scissors, Paper, respectively in one game.   
   (a) Find the joint PMF of X, Y, Z.   
   (b) Find the probability that the game is decisive. Simplify your answer (it should not involve a sum of many terms).   
   (c) What is the probability that the game is decisive for n = 5? What is the limiting probability that a game is decisive as n ! 1? Explain briefly why your answer makes sense.

Solution

Let's go through the problem step by step:

### Part (a): Joint PMF of \(X\), \(Y\), \(Z\)

Given that each of the \(n\) players independently and randomly chooses Rock, Scissors, or Paper with equal probability, we define:

- \(X\): The number of players who choose Rock.

- \(Y\): The number of players who choose Scissors.

- \(Z\): The number of players who choose Paper.

Since each player has an equal probability of choosing any one of Rock, Scissors, or Paper, the joint distribution of \(X\), \(Y\), and \(Z\) follows a multinomial distribution:

\[

P(X = x, Y = y, Z = z) = \frac{n!}{x!y!z!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^z = \frac{n!}{x!y!z!} \left(\frac{1}{3}\right)^n

\]

where \(x + y + z = n\) and \(x, y, z \geq 0\).

### Part (b): Probability that the game is decisive

The game is decisive if exactly two of the three choices appear. This means that one of \(X\), \(Y\), or \(Z\) is zero, and the other two are nonzero.

For each scenario, consider the probability:

1. \(X = 0\) (no one chooses Rock), then \(Y = y\) and \(Z = z\) where \(y + z = n\).

\[

P(X = 0, Y = y, Z = z) = \frac{n!}{0!y!z!} \left(\frac{1}{3}\right)^0 \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^z = \frac{n!}{y!z!} \left(\frac{1}{3}\right)^n

\]

The number of ways to choose \(y\) and \(z\) is given by \(\binom{n}{y}\).

2. \(Y = 0\) (no one chooses Scissors), then \(X = x\) and \(Z = z\) where \(x + z = n\).

\[

P(X = x, Y = 0, Z = z) = \frac{n!}{x!0!z!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^0 \left(\frac{1}{3}\right)^z = \frac{n!}{x!z!} \left(\frac{1}{3}\right)^n

\]

The number of ways to choose \(x\) and \(z\) is given by \(\binom{n}{x}\).

3. \(Z = 0\) (no one chooses Paper), then \(X = x\) and \(Y = y\) where \(x + y = n\).

\[

P(X = x, Y = y, Z = 0) = \frac{n!}{x!y!0!} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^y \left(\frac{1}{3}\right)^0 = \frac{n!}{x!y!} \left(\frac{1}{3}\right)^n

\]

The number of ways to choose \(x\) and \(y\) is given by \(\binom{n}{x}\).

Summing these probabilities across all valid choices gives us the total probability that the game is decisive:

\[

P(\text{decisive}) = 3 \times \sum\_{k=1}^{n-1} \frac{n!}{k!(n-k)!} \left(\frac{1}{3}\right)^n = 3 \times \left(\frac{2^n - 2}{2^n}\right) \times \left(\frac{1}{3}\right)^n

\]

Simplifying:

\[

P(\text{decisive}) = \frac{2^{n+1} - 2}{2^n} \times \left(\frac{1}{3}\right)^n

\]

### Part (c): Probability for specific cases

#### \(n = 5\):

Substitute \(n = 5\) into the expression obtained above:

\[

P(\text{decisive}) = \frac{2^{5+1} - 2}{2^5} \times \left(\frac{1}{3}\right)^5 = \frac{64 - 2}{32} \times \left(\frac{1}{3}\right)^5 = \frac{62}{32} \times \left(\frac{1}{3}\right)^5

\]

\[

P(\text{decisive}) = \frac{31}{16} \times \left(\frac{1}{3}\right)^5 = \frac{31}{16} \times \frac{1}{243} = \frac{31}{3888}

\]

#### Limiting case \(n \to \infty\):

As \(n \to \infty\), the probability that exactly two out of the three choices appear tends to zero. The intuition is that with a large number of players, it's more likely that all three choices will appear, making the game indecisive. Thus, the limiting probability that the game is decisive is:

\[

\lim\_{n \to \infty} P(\text{decisive}) = 0

\]

### Conclusion

For large \(n\), the probability of the game being decisive tends to zero. This makes sense because as the number of players increases, it's more likely that all three choices will be represented, resulting in an indecisive game.